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Chaotic Motion of a Rigid Rotator

F. M. El Sabaa[1]

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1. INTRODUCTION

The dynamical system with two degrees of freedom has been studied by many authors in the attempt to get a complete picture of the behavior of the system. It is well known that if the system possesses a first integral besides the integral of energy, then the system is completely integrable. Following the fundamental works of Kolmogorov (1954), Arnold (1963), and Moser (1973) (KAM), many theoretical and numerical results have been presented by authors who have studied the problem when the system has only the integral of energy (Jacobi's integral). The results of KAM have clarified the picture of nonintegrable systems through small perturbations of integrable systems; for small perturbations we get very regular orbits, lying apparently on invariant tori, while for larger perturbations a part of the tori seems to be destroyed and erratic orbits appear instead filling the so-called stochastic region.

On the other hand, the problem of a heavy rigid body rotating about a fixed point has not been solved except in three cases where the mass distributions satisfy certain relations. These cases are those of Euler, Lagrange, and Kovaleveskaya. Euler's (1758) case was reduced to quadrature in elliptic functions. Deprit (1967), by his variables, reduced the problem to only one degree of freedom. This reduction permits the representation of all possible solutions of the problem by isoenergetic curves in the phase plane. The perturbation of Euler's case in Deprit's variables has been treated by many authors to prove the existence of periodic solutions with small parameters in the gravity or the Newtonian

[1]Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt.

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field (see, for example, (Barkin and Ievlev, 1977; Demin and Kiselev, 1974; El-Sabaa, 1989*a*)). In the Lagrange (1888) case the problem was integrated in terms of elliptic functions, but no one has simplified this problem as for Euler's case, while in the Kovalevskaya case the problem was integrated in terms of the Riemann θ-functions of two variables, which is a very complicated solution. After the solution of Kovaleveskaya, there have been many works concerned almost entirely with the consideration of special cases, starting with the work of Applerot (1893), and including the work of Kozlov (1980) and El-Sabaa (1989*b*), attempting to find enough special cases to be able to know more about the general behavior of the problem.

In the present work we transform the equations of motion of a heavy rigid body into dynamical systems of two degrees of freedom by using isothermal coordinates. The new system has Jacobi's integral and we investigate numerically the existence of the second integral. This integral occurs when the regular orbits lie on invariant tori in the phase space. Stochastic regions indicate that the system is notintegrable.

2. THE EQUATIONS OF MOTION

Consider a set of Cartesian coordinates $OXYZ$, fixed to a rigid body with respect to a reference system $Oxyz$ fixed in the inertial space. The moving system $Oxyz$ is chosen such that the axes are directed along the principal axes of inertia for point O. The orientation of the fixed system relative to the moving one is specified by means of the Eulerian angles θ, ϕ, and ψ. The unit vector $\boldsymbol{\gamma}$ along the axes of symmetry of the body has the components γ_1, γ_2, and γ_3 connected with Eulerian angles by the relations

$$\gamma_1 = \sin\theta \sin\phi, \qquad \gamma_2 = \sin\theta \cos\phi, \qquad \gamma_3 = \cos\theta \tag{1}$$

while the components of the angular velocity of the body $\boldsymbol{\omega}$ can be expressed in terms of Eulerian angles and of their temporal derivatives as follows:

$$\begin{aligned} p &= \dot{\psi} \sin\theta + \dot{\theta} \cos\phi \\ q &= \dot{\psi} \sin\theta \cos\phi - \dot{\theta} \sin\phi \\ r &= \dot{\psi} \cos\theta + \phi \end{aligned} \tag{2}$$

The Lagrangian function of the system can be written in the form

$$L(\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi})$$

$$= \frac{1}{2}[A(\dot{\psi} \sin\theta \sin\phi + \theta \cos\phi)^2$$

$$+ B(\dot{\phi} \sin\theta \cos\phi + \dot{\theta} \sin\phi)^2 - C(\dot{\psi} \cos\theta + \dot{\phi})^2] - V(\theta, \phi)] \qquad (3)$$

where A, B, and C are the principal moments of inertia, and V is the potential energy defined as

$$V(\theta, \phi) = mg(X_0 \sin\theta \sin\phi + Y_0 \sin\theta \cos\phi + Z_0 \cos\theta) \qquad (4)$$

where X_0, Y_0, and Z_0 are the components of the radius vector of the center of mass in the reference system which is fixed to the body, m is the mass of the body, and g is the acceleration due to gravity.

The equations of motion are

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \\ &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \\ &\left(\frac{\partial L}{\partial \dot{\psi}}\right) = f \end{aligned} \qquad (5)$$

where ψ is a negligible coordinate. The system (5) is a nonlinear system of differential equations in the unknown functions $(\theta, \phi, \psi, \dot{\theta}, \dot{\phi})$. This system can be solved if there exist four time-independent integrals. There are three known cases where the problem can be solved—Euler, Lagrange, and Kovalevskaya—where the mass distribution satisfies certain relations. In general, the problem of a rigid body has not been solved and is in a sense unsolvable.

3. THE REDUCTION OF THE EQUATIONS

Aside from the three cases mentioned above, many other cases are known for solving the rigid-body problem either by assuming some restrictions on the constants of integration or by transforming the equations of motion into a more easily reduced system. According to Yahya (1976), the system of equations (5) can be transformed into a system of two degrees of freedom using isothermal coordinates as follows:

The Routhian function of the system is written as

$$R=\frac{D}{2}\left\{C(A\sin^2\phi+B\cos^2\phi)\sin^2\theta\dot{\phi}^2-\frac{1}{2}C(A-B)\sin 2\theta\sin 2\phi\dot{\theta}\dot{\phi}\right.$$
$$\left.+\left[\frac{1}{D}(A\cos^2\phi+B\sin^2\phi)-(A-B)^2\sin^2\theta\sin^2\phi\cos^2\phi\right]\dot{\theta}^2\right\}$$
$$+fD\sin\phi\cos\phi\cos\theta\frac{d}{dt}[C\ln|\tan\phi|-(A-B)\ln|\cos\theta|]+V(\theta,\phi)$$
$$-\frac{1}{2}f^2D \tag{6}$$

where

$$D=(A\sin^2\phi+B\cos^2\phi)\sin^2\phi+C\cos^2\theta \tag{7}$$

If we introduce the new variables ξ, η, and ζ related to the Eulerian angles by the relations

$$\sin\theta\sin\phi=\sqrt{A}\xi,\qquad \sin\theta\cos\phi=\sqrt{B}\eta,\qquad \cos\theta=\sqrt{C}\xi \tag{8}$$

then (ξ,η,ζ) are the coordinates of any point on the surface of ellipsoid inertia

$$A\xi^2+B\eta^2+C\zeta^2=1 \tag{9}$$

The Routhian function becomes

$$R=\frac{1}{2}ABCD(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)+\frac{fD(ABC)^{1/2}}{1-C\zeta^2}$$
$$\times[C\zeta(\eta\dot{\xi}-\xi\dot{\eta})-(A-B)\zeta\eta\dot{\xi}]+V-\frac{1}{2}f^2D \tag{10}$$

Let

$$(\xi,\eta,\zeta)$$
$$=\frac{1}{[(1-n\alpha^2)(1+m\rho^2)]^{1/2}}$$
$$\times\left[\frac{\sqrt{B}}{A}\alpha(1-k'^2\rho^2)^{1/2},\quad \frac{1}{\sqrt{B}}\{(1-\alpha^2)(1-\rho^2)\}^{1/2},\quad \frac{\sqrt{B}}{C}\rho(1-k^2\alpha^2)^{1/2}\right]$$

where

$$k^2=1-k'^2=\frac{A-B}{A-C},\qquad n=\frac{A-B}{A},\qquad m\frac{B-C}{C} \tag{11}$$

The quadratic terms of the Routhian function in the old velocities $\dot{\xi}$, $\dot{\eta}$, and $\dot{\zeta}$,

$$R_2 = \frac{1}{2} ABCD(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)$$

are transformed into quadratic terms in the new velocities $\dot{\alpha}$ and $\dot{\rho}$ as follows

$$R_2 = \frac{B\aleph}{2}\left[\frac{C}{A}\frac{\dot{\alpha}^2}{(1-n\alpha^2)^2(1-k^2\alpha^2)(1-\alpha^2)} + \frac{A}{C}\frac{\dot{\rho}^2}{(1+m\rho^2)^2(1-k'^2\rho^2)(1-\rho^2)}\right] \tag{12}$$

where $\aleph = 1 - k^2\alpha^2 - k'^2\rho^2$.

Introducing new variables x, y such that

$$x = \left(\frac{C}{A}\right)^{1/2} \int_0^{\alpha} \frac{d\alpha}{(1-n\alpha^2)[(1-k^2\alpha^2)(1-\alpha^2)]^{1/2}} \tag{13}$$

$$y = \left(\frac{A}{C}\right)^{1/2} \int_0^{\alpha} \frac{d\rho}{(1+m\rho^2)[(1-k'^2\rho^2)(1-\rho^2)]^{1/2}} \tag{14}$$

with $d\tau = dt/B\aleph$, we find that the Routhian function takes the form

$$R = \frac{1}{2}(x'^2 + y'^2) + \frac{f}{M}(PTx' - QSy') + U$$

where

$$\begin{aligned}
P &= (1-n\alpha^2)^{1/2}\left(1 - \frac{A+B-C}{A}k^2\alpha^2\right)\\
Q &= n(1+m\rho^2)^{1/2}\left(1 - \frac{A+B-C}{C}\rho^2\right)\\
S &= \alpha[(1-\alpha^2)(1-k^2\alpha^2)(1-n\alpha^2)]^{1/2}\\
T &= \rho[(1-\rho^2)(1-k'^2\rho^2)(1+m\rho^2)]^{1/2}\\
M &= 1 - n\alpha^2 - \rho^2 + \left(1 - \frac{B}{A}k'^2\right)\alpha^2\rho^2
\end{aligned} \tag{15}$$

The equations of motion in the new variables are

$$x'' + \Omega y' = \frac{\partial U}{\partial x'}, \qquad y'' - \Omega x' = \frac{\partial U}{\partial y} \qquad \left(' \equiv \frac{d}{d\tau}\right) \tag{16}$$

The system (16) is a plane motion system of two degrees of freedom under the action of a potential and gyroscopic force U and Ω defined by

$$U = B\aleph\left[E + V - \frac{f^2}{2B}(1 - n\alpha^2)(1 + m\rho^2)\right] \tag{17}$$

$$\begin{aligned}\Omega = {} & \frac{f\aleph}{(AC)^{1/2}}[(1 - n\alpha^2)(1 + m\rho^2)]^{1/2} \\ & \times [A - B + C - 2(A - B)\alpha^2 + 2(B - C)\rho^2]\end{aligned} \tag{18}$$

and possesses the Jacobi integral

$$x'^2 + y'^2 = 2U \tag{19}$$

4. THE PERIODIC ORBITS OF THE MOTION

The nonintegrable system (16) is quasiintegrable in the sense that there exist invariant tori near the stable periodic solutions, in accordance with KAM theory, which states that almost all the invariant tori of the unperturbed system remain in spite of the presence of a small perturbation. The system (16) was solved analytically to get the periodic solutions about the equilibrium positions for small perturbation. The results are compatible with the KAM theory, since the invariant tori exist in the neighborhood of the stable periodic solution (Lyapunov, 1966).

Hénon and Heiles (1964) used computer calculations to show that the invariant tori exist for small values of the energy constant, and consequently the system is close enough to its periodic solutions; they showed also that for a large value of the energy constant (escape energy), stochastic regions appears instead of the invariant tori.

We used the fixed-point method introduced first by Poincaré (1975) and continued by Birkhoff (1927) and Moser (1973) to obtain the invariant curve of the system (16). The solution of (16) represents the trajectory in the phase space (x, x', y, y') and along this trajectory the value of the Jacobi constant is fixed. Thus, for a given value of this constant, the trajectory of the problem will be treated in three-dimensional space (x, y, x'). Let us examine the consecutive crossing of this trajectory with the plane (x, x') in the positive direction, i.e., points of the trajectory which satisfy the conditions $y = 0$, $y' > 0$ in the phase plane (x, y). These points correspond to the crossing of the trajectory of the point $x = x(\tau)$, $y = y(\tau)$.

The investigation of this trajectory is reduced to the study of the manifold of such found points in the phase plane (x, x).

For the initial conditions $x = x_0$, $y = 0$, $x' = 0$, and y' obtained from the Jacobi integral

$$y'^2 = 2B\aleph\left[E + V - \frac{f^2}{2B}(1 - n\alpha^2)(1 + m\rho^2)\right] \tag{20}$$

and a fixed value for the constant E, equations (16) can be integrated numerically.

The points of intersection are always inside the curve representing the periodic orbits $y = y' = 0$.

If the curve is closed, then all other orbits will lie inside this curve and then we have an ordered motion. If the curve is open, then the orbits will be on one side of the curve and then we have chaotic motion.

5. THE ANALYSIS OF THE EQUATIONS OF MOTION

To study the perturbation of integrable cases of the rigid-body problem, it is required to seek the initial conditions of the system (16) that correspond to the integrable cases of the rigid-body problem.

For $\aleph = 0$ we have the ellipse

$$\frac{\alpha^2}{(1/\sqrt{k})^2} + \frac{\rho^2}{(1/\sqrt{k'})^2} = 1, \qquad 0 \leq k, k' \leq 1$$

Then for $\aleph > 0$ there exist α, ρ such that $-1 < \alpha$, $\rho < 1$. So if $k = 0$ and $k' = 1$, then $\aleph = 1 - \rho^2 > 0$ and if $k = 1$ and $k' = 0$, then $\aleph = 1 - \alpha^2 > 0$. According to Jacobi's integral (19) for Euler's case ($X_0 = Y_0 = Z_0 = 0$), we have either

$$0 \leq n < 1 \qquad \text{and} \qquad m \leq 0$$

or

$$n \leq 0 \qquad \text{and} \qquad 0 \leq m \leq 1$$

and then we get

$$E - \frac{f^2}{2B}(1 - n\alpha^2)(1 + m\rho^2) > 0 \tag{21}$$

In the case of $f = 0$, the constant $E > 0$.

We consider the following cases for the moments of inertia A, B, and C.

1. $A = B > C$. In this case $k = 0$, $k' = 1$, $n = 0$, and $m > 0$, the relation (21) becomes

$$E > \frac{f^2}{2B}(1 + m\rho^2) \tag{22}$$

So for any value of α, the quantity ρ is restricted by the condition

$$\rho^2 > \frac{(2BE/f^2) - 1}{m} \quad \text{with} \quad E > \frac{f^2}{2B} \tag{23}$$

2. $B = C > A$. By the same procedure, we choose the initial values of ρ and α such that

$$\alpha^2 > \frac{(2BE/f^2) - 1}{n} \quad \text{with} \quad E > \frac{f^2}{2B} \tag{24}$$

and ρ takes any value.

3. $A = B < C$. In this case, we have $k = 0$, $k' = 1$, $n = 0$, and $0 < m < 1$, and then we get

$$\rho^2 > \frac{(2BE/f^2) - 1}{m} \quad \text{with} \quad E > \frac{f^2}{2B}(1 + m) \tag{25}$$

with any value of α.

4. $A > B = C$. This case is similar to the one when ρ takes any value and α is restricted by the condition

$$\alpha^2 > \frac{1 - 2BE/f^2}{n} \quad \text{with} \quad E > \frac{f^2}{2B}(1 - n) \tag{26}$$

where $0 < n < 1$ and $m = 0$.

5. $A > B > C$. We have

$$(1 - n\alpha^2)(1 + m\rho^2) < \frac{2BE}{f^2} \tag{27}$$

where $0 < k < 1$, $0 < k' < 1$, $0 < n < 1$, and $m > 0$, so that $0 < 1 - n\alpha^2 \leq 1$ and $1 + m\rho^2 \geq 1$, which leads to two cases

$$1 - n\alpha^2 < \frac{2BE}{f} \quad \text{with} \quad \frac{2BE}{f^2} < 1$$

and

$$(1 - n\alpha^2)(1 + m\rho^2) < \frac{2BE}{f^2} \quad \text{with} \quad \frac{2BE}{f^2} > 1$$

Figure 1 shows the invariant curves for small perturbation of Euler's case ($X_0 = 0$, $Y_0 = 0.2$, $Z_0 = 0.02$). The invariant curves are closed. This means that the motion is ordered.

The computer calculations show that the invariant curves do not depend on the increase of E, where the Poincaré mappings lie on the invariant tori for large values of E.

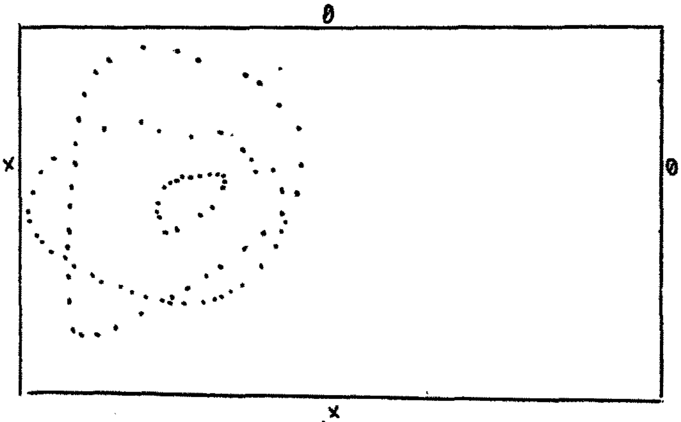


Fig. 1. The invariant curves for small perturbation of the Euler case ($X_0 = 0$, $Y_0 = 0.2$, $Z_0 = 0.02$) for different value of E: 0.5, 1.01, 1.5.

For a large perturbation, part of the tori is destroyed and the orbits lie on one side of the curve, which means that chaotic motion appears in this case. Figure 2 shows this case when the center of mass takes the values (0, 1, 1.5) and (2, 2, 3), respectively. Figures 3 and 4 show the orbits in the xy plane corresponding to these cases, while Fig. 5 shows the periodic orbit

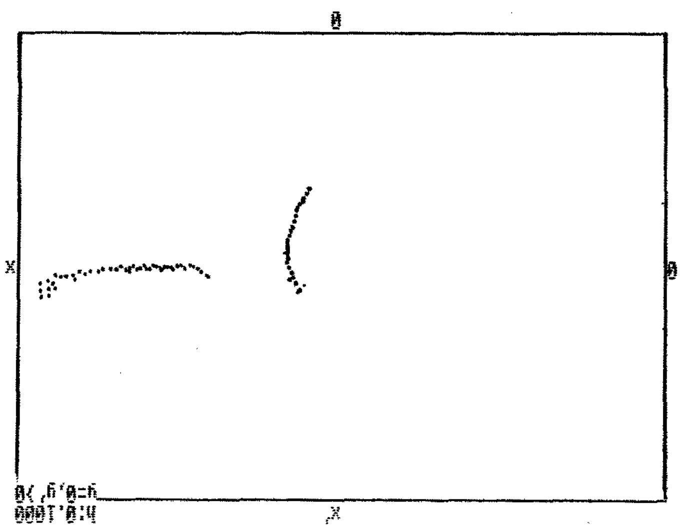


Fig. 2. Chaotic motion appears instead of regular motion for large perturbations of the Euler case.

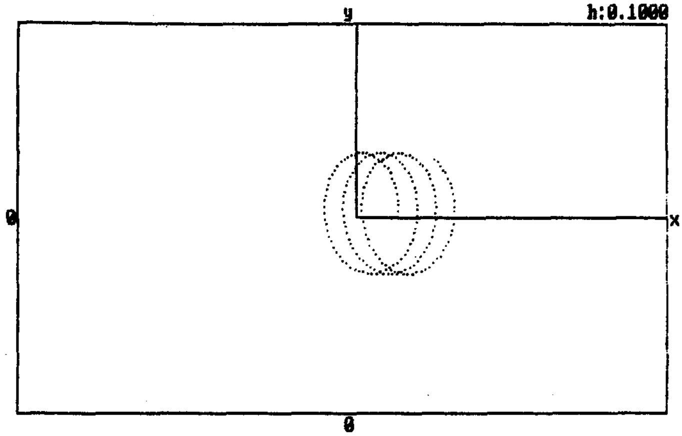


Fig. 3. The orbit in the xy plane for large perturbation of the Euler case where the center of mass is (0, 1, 1.5).

of the small perturbation of Euler's case. The small perturbation of Lagrange's case can be taken as a continuation to study the regular and chaotic motion of a rigid-body problem.

Figure 6 shows the several orbits which have been computed, and for each of them the corresponding points appear to lie on a regular closed curve, which means again that the motion is ordered.

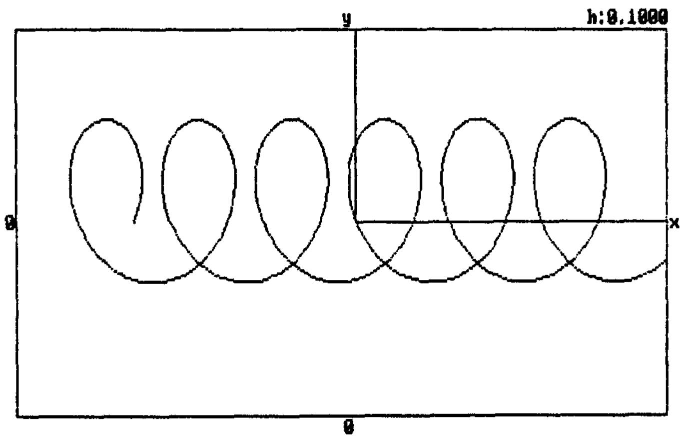


Fig. 4. The orbit in the xy plane for large perturbation of the Euler case when the center of mass is (2, 2, 3).

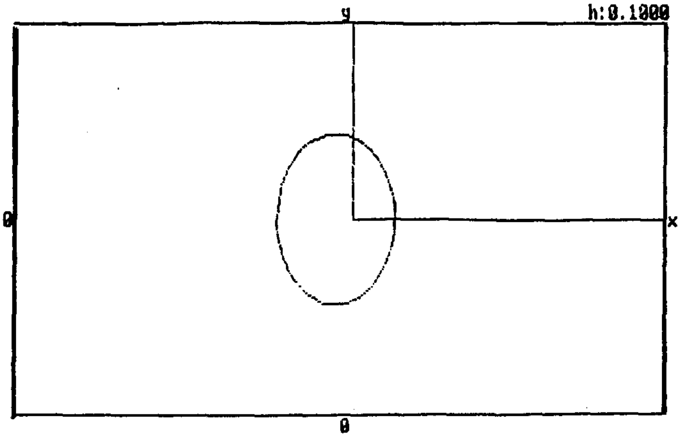


Fig. 5. The closed orbit for small perturbation of the Lagrange case.

Figures 7–9 show some orbits in the xy plane corresponding to the regular motion. Figure 10 represents the chaotic motion for a large perturbation of Lagrange's case, $A - B = 5$, $X_0 = 2$, $Z_0 = 3$, for several values of α and x'.

The case of the Kovaleveskaya problem has been studied in detail in El Sabaa (n.d.) by using Kolossoff's transformation, which transforms the

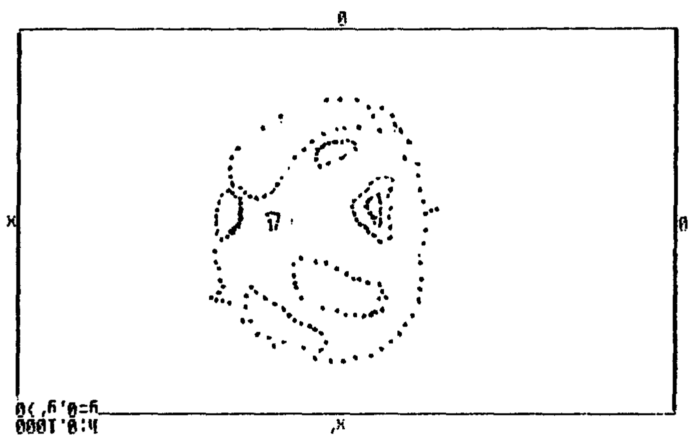


Fig. 6. The regular motion for small perturbation of the Lagrange case where $A - B = 0.02$, $X_0 = 0.01$, $Y_0 = 0.02$, $Z_0 = 1$ for different values of α and x'.

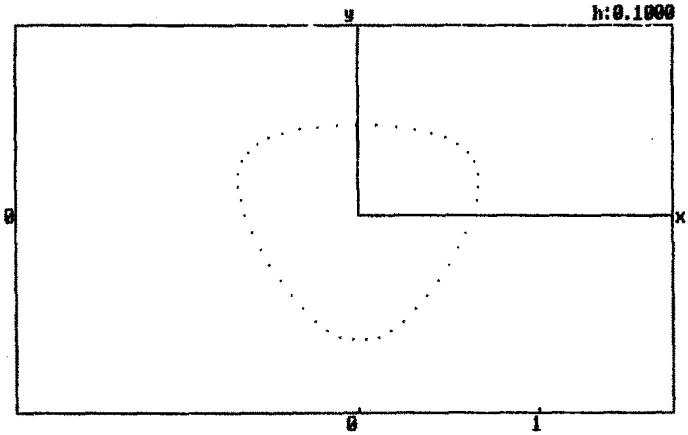


Fig. 7. The closed orbit for small perturbation of the Lagrange case.

problem into a plane particle motion under a certain potential function and then the method of surface section is used.

6. CONCLUSIONS

We studied regular and chaotic motion in the problem of a heavy rigid body with a fixed point. The results agree with the KAM theorem. Here we

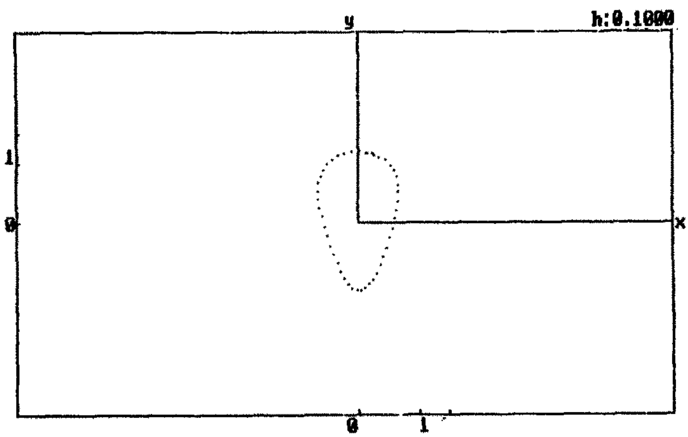


Fig. 8. The closed orbit for small perturbation of the Lagrange case.

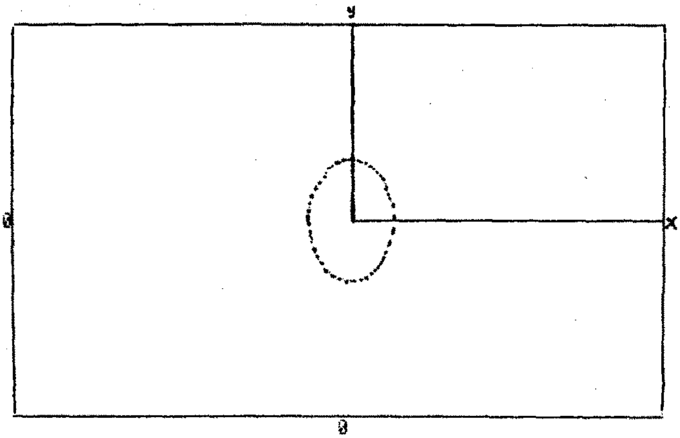


Fig. 9. The closed orbit for small perturbation of the Lagrange case.

used the method of surface section after transforming the equation of motion of a rigid body into a system of two degrees of freedom with gyroscopic force. This force represents a great difficulty in the numerical calculations. Moreover, there are restrictions on the initial conditions where $A \neq C$ and $A \neq 0$ and $C \neq 0$. In the ordered motion with a convenient perturbation, the invariant curves remain closed but their shape changes according to the change of initial conditions of α and x'.

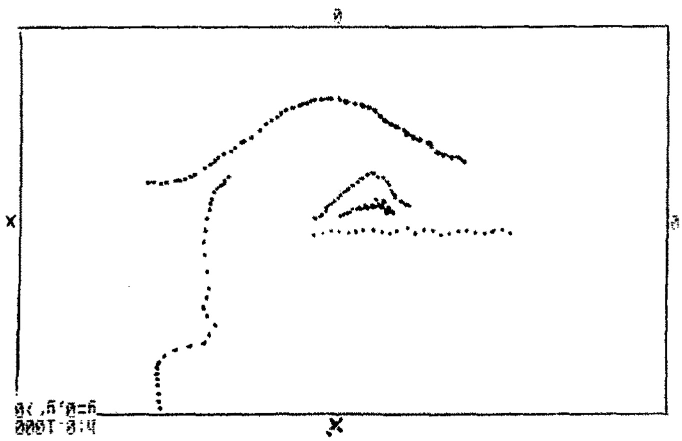
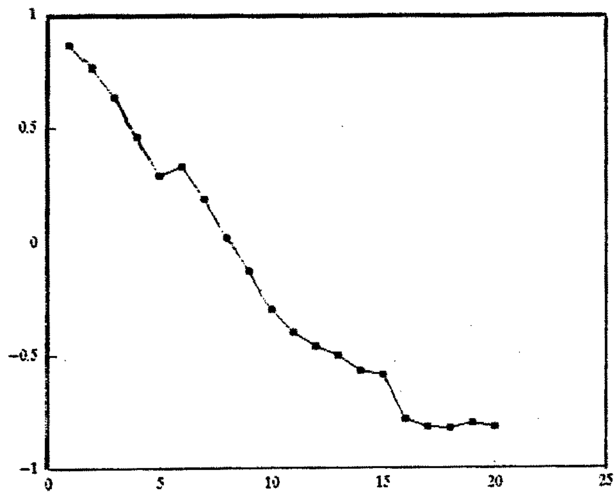


Fig. 10. The chaotic motion of the Lagrange case for large perturbation.

This work is confirmed by calculating the correlation function $C(s)$ of some mapped points.

If (x_i, x_i') are the coordinates of the i image point, $i = 1, 2, \ldots, n$, then the correlation function is defined as

$$C(s) = \frac{\sum_{i=1}^{n-s} (x_i' - \langle x' \rangle)(x_{i+s}' - \langle x' \rangle)}{\sum_{i=1}^{n-s} (x_i' - \langle x' \rangle)^2}$$



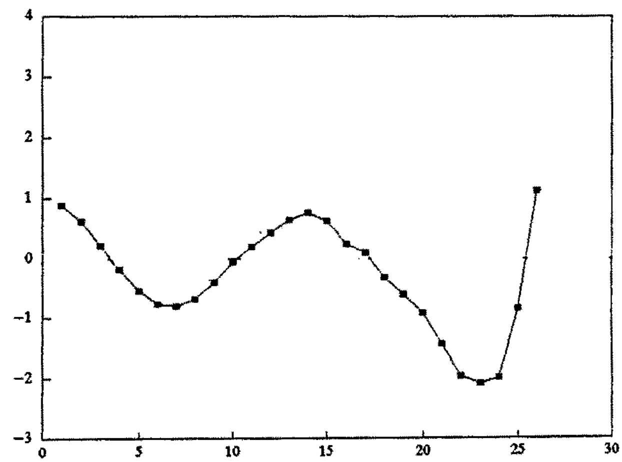


Fig. 11. Correlation functions $C(s)$ for initial points on stochastic and regular regions.

where

$$\langle x' \rangle = \frac{1}{n} \sum_{i=1}^{n} x'_i$$

We take the Euler case as an example to see the dispersity of the mapped points.

As shown in Fig. 11, $C(s)$ decays rapidly for the mapped points in the stochastic region, while it oscillates for the points in the regular region.

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